A version of this course has been Approved for the new B.Ed. program at York:

Hands-on Mathematics to support Interdisciplinary Learning

Walter Whiteley whiteley@mathstat.yorku.ca

Brief Course Description:

Mathematics learning at the K-8 level is best supported with multiple representations and with hands-on activities. This course will offer opportunities for future PJ and JI teachers to explore these key mathematical processes, identified in the Ontario curriculum, through investigations, reflections, and selective readings from the literature.
Many pre-service teachers have not themselves experienced learning and doing mathematics with these broader, more connected and inclusive approaches. They lack the confidence and the background resources to support these mathematical processes and approaches with their students. Reworking sample mathematics with these supports, including manipulatives (as described in the Ontario Curriculum) and appropriate technology (dynamic geometry, as identified in the Ontario Curriculum) provides opportunities for future teachers to develop their mathematical repertoire and confidence, as well as their capacity to offer active engagement with multiple approaches that can support the diversity of student thinking found in problem solving situations.
The activities selected, and the associated group projects, will also cross curricular boundaries to connect mathematics and spatial reasoning to learning objectives and concepts from science and arts, supporting a broad range of connected student learning in the STEM and STEAM disciplines.
A key component will be building and exploring embodied approaches, including spatial reasoning, kinesthetic and spatial-visual representations to support learning and thinking: for children and in the elementary classrooms. These approaches are also central in the practices of mathematics, the sciences and many of the arts, at all levels and over a lifetime of student and teacher learning.

Expanded Course Description:

The overall purpose of this course is two-fold. First, it is to provide elementary teachers with an opportunity to interact with areas of mathematics, processes of mathematics and diverse approaches to doing mathematics that they may not initially feel confident with. Particularly, pre-service teachers will be given a chance to explore the skills and conceptual foundations of spatial reasoning, the importance for spatial reasoning in the elementary curriculum and how spatial reasoning and hands on mathematical activities can be used opportunity and playground to make multiple cross-curricular links.
Second, students in our classrooms are central to the education profession and preparing to equip students with spatial reasoning skills will have both short term and long-term benefits. Spatial reasoning is a skill that can be taught and nurtured with practice and the short-term effects include providing students with another method of interacting with mathematics. Studies have shown that, for many students, spatial reasoning is integral to building student confidence in mathematics as it provides multiple perspectives in problem solving and inquiry. Children enter school after years of spatial (3D) activity. It is critical that the classroom build on the abilities developed and connect these to curriculum objectives and classroom activities which too easily focus on 2d, desk and paper-based representations and assessments.
Mathematics can appear dull, unappealing and even intimidating to many students. If teachers become experienced in spatializing the curriculum and the classroom, with hands on activities, they can offer a more inclusive classroom that supports the diversity of student prior knowledge, motivation, and ways of problem solving. Such a classroom offers multiple ways that mathematics can be recognized in real world experiences and in other subjects and brought into the mathematics classroom. Viewing mathematics through this key additional lens can appeal to the visual and kinesthetic

abilities of students who may not no otherwise be given the chance to express and engage their skills within forms of mathematics that are devoid of spatial representations and reasoning. Spatializing the curriculum can also make mathematics more enjoyable for a wide range of students as they discover the ways that mathematics can be recognized and used across the sciences and social sciences. Additionally, teachers can use spatial reasoning to their advantage by connecting to topics in multiple strands and subject areas in integrated lessons within an elementary classroom.
Critical to learning mathematical concepts is developing a flexible range of examples, including what 'changes' create other examples, and a flexible range of non-examples and what change create other non-examples. The explorations will consistently play with developing the pre-service teachers capacity to play with the examples, to ask 'what if …' and consider how they can develop answers to their own (and student) questions.
In a particular year of this course, key topics will likely be drawn from the following list, chosen from the Elementary Mathematics Curriculum, along with topics raised by the students and their questions,

1 Proportional Reasoning: (Considering shape, measurement and numeracy in relative terms, and with ratios).

- Filling the pyramid activity
  - Using sand or water to fill a pyramid and have students make educated guesses on the half-way, quarter-way mark etc., and compare 'similar shapes of the top surface
- GSP Dilation Sketch
  - Comparing two similar shapes whose only differences is a difference of scale, and connect to filling activity
- Number line explorations, including transformations.
- Measurement / change of scale (including Converting units).
- Angles including angle measures as ratios of whole turns

2. Changing Dimension, including Area and volume.

- Length: linear (1 dimension) scale by k
- Area: 2-dimension: scale by k^2
- Volume: 3-dimension: scale by k^3
- Possibly Fractals (Mathematical Set composed of repeating patterns at every scale (zooming in or out). This is an art-mathematics connection.
- representing 3D objects in 2D, and identifying 3D objects from various 2D representations.

3 Transforming / Symmetry

- Mental rotation, and 3D explorations of transforming perspectives on 3D objects
- reflection / rotation in the plane and space (including work with mirrors, MIRA)
  - Several activities that utilize the GSP program can be used. Whole lessons can be centered on use of GSP sketchpad that allows students (and teachers) to thoroughly explore reflections and rotations and the properties of these types of transformations,
- Composition of transformations (group operations)
  - Combinations of reflections and/or rotations can yield different isometries (distance preserving maps), including translations. GSP activities will offer future elementary teachers opportunities to explore these compositions and readings will explore what students can provide to be used in the classroom.
- Symmetries of the number line, including recent research showing symmetry improves learning of negative numbers.
- Tilings, frieze patterns, rosettes and related plane patterns and their transformations (connected to art and to a range of examples from many cultures, including aboriginal designs). Using web-based and app based tiling programs (e.g. i-Ornament) for exploration
- Quadrilateral symmetry and classifications
  - Elementary Teachers are given the opportunity to become more familiar with the symmetric properties and classifications of quadrilaterals and other polygons, including explorations of inclusive definitions and associated local reasoning.

4. Polyhedra / Polygons (using Polyedron, other building materials, web based programs).

- Euler's Formula ($|V|-|E|+|F|=2$): when this works and when it fails.
- Platonic Solids and what shapes make fair dice.

5. Growing plane and Patterns in the pre-algebra strand, through rules and functions.

Key course objectives/UUDLEs include the development of:

1) an awareness of the limits of knowledge and personal responsibility, including supporting a growth mindset (everyone can learn mathematics and I can learn mathematics) within their mathematical communities of practice.
2) an understanding of ways of knowing and how mathematical knowledge is made, learned, and used, with an awareness of connections to interdisciplinary knowledge;
3) the capacity to engage meaningfully with questions of curriculum, perspective, and the

dynamics of learning, with an emphasis on making sense through multiple representations, investigations, spatial reasoning and 'seeing geometry everywhere';
4) critical engagement with Ontario curriculum and policy documents in mathematics and in other disciplines;
5) the ability to observe, discern, critique, assess and act accordingly when doing mathematics individually and in communities of practice, including asking mathematical questions embedded in 'making sense' of the mathematical observations.
6) the ability to articulate curricular and pedagogical intent when making connections and developing investigations in hands on mathematics.
7) an interest in how spatial sense-making and representations are found across history and across cultures, as well as the individual diversity in this sense-making.

Course Design

The course may be offered in a face-to-face or blended format. The course design supports students in achieving the learning objectives through the development of group projects, the use of a wide variety of technologies, and the use of manipulates to support collaborative learning, learning use of hands, eyes and movement, and continuing reflection. The design of the course permits feedback and peer-to-peer exchanges that support multiple approaches to investigations, and learning with safe opportunities for mistakes, partial insights, and developing conceptual networks
With support of York Ethics Approvals, the classroom investigations will be supported by participation of M.Ed. students who are also engaged in documenting the impact of these activities, and in research on spatial reasoning. The initial course design was sketched out by three Undergraduate Students as a 4th year project course in Mathematics, and we hope to continue to include projects of final year math / concurrent education students in developing, refining and integrating hands on activities.

Tentative list of components of assessment.

Students will be expected to actively participate in Weekly mathematical investigations, regular readings, and reflective in-class discussions.
This work will lead to
- participation in investigations and class discussions (10 %) (Course objectives #1, 2, 3, 4, 5);
- regular Assignments, developing the mathematical representations and connections observed or developed within the group investigations (20%) (Course objectives #1, 2, 3, 4, 5);
- Journals Entries describing how their experiences and perspectives on mathematics are evolving, and how the hands-on activities support their growth mindset and the development of a community of practice (20%) (Course objectives #1, 2, 3, 5, 6, 7):
- Reading Responses (20%) (Course objectives #1, 2, 3, 4, 5):

Final Group Integrative Project: lesson plan for hands-on classroom activity, with links to curriculum in several disciplines and to spatial reasoning, with in-class sample investigation (30 %) (Course objectives #1, 3, 4, 5, 6, 7).

Bibliography

Resource Book:

Davis, Brent and the Spatial Reasoning Study Group (2015): Spatial Reasoning in the Early Years: Principles, Assertions, and Speculations, Routledge, ISBN-10: 1138792047

Additional background books

Jo Boaler (2016): Mathematical Mindsets: Unleashing Students' Potential through Creative Math, Inspiring Messages and Innovative Teaching, Josey-Bass

Middle Grades Mathematics Project: Spatial Visualizations (Mary Jean Winter, Glenda Lappan, Elizabeth Philips, and William Fitzgerald) (1986) Addison Wesley,

Resource Articles and Bibliography:

Ellis-Davies, Arthur (1986): Symmetry in the Mathematics Curriculum, *Mathematics in School*, The Mathematics Association, Vol. 15, No. 3 (May, 1986), pp. 27-Stable URL: http://www.jstor.org/stable/30214083 .

Paying Attention to Spatial Reasoning: Support document for Paying Attention for Mathematics Education, LNS: https://www.edu.gov.on.ca/eng/literacynumeracy/LNSPayingAttention.pdf

Moss, J., Bruce, C., Caswell, B., Hawes, Z., & Flynn, T. (in press, publication Spring 2016). *Taking Shape: Spatial Reasoning for Young Children*. Pearson Canada.

Mamolo, Ami , Margaret Sinclair, and Walter Whiteley(2011) Filling the pyramid: An activity in proportional reasoning, *Mathematics Teaching in the Middle School* (16) 2011, pages 545-551.

Sinclair, Nathalie and Catherine D. Bruce (2015) New opportunities in geometry education at the primary school, *ZDM Mathematics Education* 47:319–329, DOI 10.1007/s11858-015-0693-4

Skemp, Richard (1976) Relational Understanding and Instrumental Understanding, *Mathematics Teaching*, 77, 20–26, (1976)

Sorby, Sheryl (2009): Developing spatial cognitive skills among middle school students, *Cognitive Processing*, 2009, Volume 10 (Supp2 ICSC 2009 Special Issue), pp 312-315

The Ontario Curriculum Grades1-8 Mathematics (2005)

The Ontario Curriculum Grades 1-8 Science and Technology (2007)

The Ontario Curriculum Grades 1 to 6 Social Studies, Grades 7 and 8 History and Geography (2013): Section on Spatial Skills: Using Maps, Globes, and Graphs.

The Ontario Curriculum Grades 1-8 Arts (2009)

Tsang, Jessica M. Kristen P. Blair, Laura Bofferding & Daniel L. Schwartz, (2015) Learning to "See" Less Than Nothing: Putting Perceptual Skills to Work for Learning Numerical Structure, *Cognition and Instruction* Volume 33, Issue 2, pages 154-197 DOI: 10.1080/07370008.2015.1038539

Web Sites

Spatial Reasoning Mathstat Wiki Pages (Walter Whiteley):
http://wiki.math.yorku.ca/index.php/CMEF_Early_Geometry
http://wiki.math.yorku.ca/index.php/CMEF_Geometry_Curriculum

Ontario Association of Mathematics Educators: http://www.oame.on.ca/main/index1.php (all York Faculty of Education pre-service teachers have members login access)

Dutepe Paksu, Asuman and Walter Whiteley (2015) OAME 2015 Materials on Classifying Quadrilaterals: http://oame2015.wikispaces.com/Day+3+Saturday+May+9+Files

Jo Boaler (MOOC) How to Learn Math: For Students | Stanford Lagunita, *https://lagunita.stanford.edu/courses/Education/EDUC115-S/Spring2014/about*

Other Resources

The Computer Lab in Mathematics (Gauss Lab) has a GSP License and lab in Education also with GSP license, as well as other applications. (Note: the Ontario Ministry of Education licenses GSP for all students and teachers in Faculties of Education and all students and teachers in publically funded schools in Ontario.)
There is a full cupboard of manipulatives (Mirra, mirrors, multiple classroom sets of Polydron) materials for activities (sand and containers for pouring, mix and frames for bubbles) and books and journals to support projects in Ross S525 (MathStat Lab). These were already acquired to support Mathematics 3052: Experiencing Geometry and the room is open almost 7 hours every weekday.
There are further manipulative resources from the Educational Resource Center, some of which were purchased to support the Spatial Reasoning research of Professors Margaret Sinclair and Walter Whiteley.
The advantages of good access to these resources suggests the class will work best if run early or late in the day in Ross S525 (the MathStat Tutorial Lab during the day) that has a maximum class size of 48, with students seated in groups of up to 6 at round tables.

Course Rationale

This course is designed to be one mathematics related elective in the two year B.Ed. program, from which PJ and JI students may choose at least one.
Hands on mathematics supports active learning by students with rich connections across all of mathematics and statistics, and with other disciplines. In particular, Spatial Reasoning has been found to be a critical skill for multiple disciplines, overall age groups. Research confirms both that children enter school with substantial experience and skills in spatial reasoning, and that teachers consistently underestimate what children will do when offered challenging activities. See for example the earlier research of M. Sinclair and W. Whiteley, and work of members of the Spatial Reasoning Group. The good news is that spatial reasoning is malleable – it can be improved and developed at all ages, but requires attention and hands on activities.
More generally, this malleability applies across mathematics – particularly through Mathematical Growth Mindsets (Boaler) that reset teachers' and students' prior conceptions that they are "not good at math" – or that "math ability is fixed" and engagement will not make a difference. The impact of mindsets, including mindsets on spatial reasoning is differential – having more impact on girls and on minority students. Key parts of this impact are through elementary teachers with a fixed rather than a growth mindset. A key finding of the mindset work and related cognitive studies is that people's brains are most active when they are making mistakes, and brains grow most when noticing mistakes. The course pedagogy will support this type of learning through investigations that include making mistakes.
The design of the course supports the use of multiple representations, and the development of connections among concepts, both key Mathematics Processes in the Ontario Mathematics Curriculum and in the broader elementary curriculum in Ontario. The course will assist future teachers to make connections across the curriculum, and develop lessons that integrate learning objectives from multiple strands and subjects.